On Kudla's Green function for signature (2,2) Part II

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Abstract

Around 2000 Kudla presented conjectures about deep relations between arithmetic intersection theory, Eisenstein series and their derivatives, and special values of Rankin $L-$series. The aim of this text is to work out the details of an old unpublished draft on the second author's attempt to prove these conjectures for the case of the product of two modular curves.

In part one we proved that the generating series of certain modified arithmetic special cycles is as predicted by Kudla's conjectures a modular form with values in the first arithmetic Chow group. Here we pair this generating series with the square of the first arithmetic Chern class of the line bundle of modular forms. Up to previously known Faltings heights of Hecke correspondences only integrals of the Green functions $\Xi(m)$ over X had to be computed. The resulting arithmetic intersection numbers turn out to be as predicted by Kudla to be strongly related to the Fourier coefficients of the derivative of the classical real analytic Eisenstein series $E_2(\tau, s)$.

Contents

1 Introduction

As in the first part of our work [BK1] we consider the natural models of product of modular curves $X = X(1) \times X(1)$ and its Hecke correspondence $T(N)$ over the integers $\mathbb{Z}$. We had introduced the modified arithmetic special cycles

$$\widehat{Z}_\rho(m) := (T(m), \widetilde{\Xi}_\rho(v, z, m)) \in \widehat{CH}^1(X)$$

and a modified Kudla generating series

$$\widehat{\phi}_{K,\rho} = \sum \widehat{Z}_\rho(m) q^m. \tag{1.0.1}$$

We proved in [BK1]that $\widehat{\phi}_{K,\rho}$ is a modular form for $\mathrm{SL}_2(\mathbf{Z})$ of weight k with coefficients in the arithmetic Chow group $\widehat{CH}^1(X)$.
Hence, for all linear maps $L : \widehat{CH}^1(X) \to \mathbb{R}$ the series $L(\widehat{Z}_\rho(m))q^m$ is a (nonholomorphic) $\mathbb{R}-$valued modular form in the usual sense. Now we denote by $\widehat{c}_1(\overline{\mathcal{L}})$ the first arithmetic Chern class[1] of the line bundle of modular forms $\overline{\mathcal{L}}(12, 12)$ of bi-weight $(12, 12)$ equipped with the Petersson metric. Then we choose the linear map $L(-) = \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot (-)$ and prove the following result, which complements the work of Kudla, Rapoport and Yang in the $O(1, 2)$ case [KRY], [KRY1] and provides a first confirmation of Kudla's conjectures [Ku] in dimension 2.

Main Theorem (modified Kudla conjecture)**.** *We have an identity of modular forms*

$$\widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{\phi}_{K,\rho} = \mathbb{E}_2'(\tau, 1) + f_\rho(\tau) \tag{1.0.2}$$

with $\mathbb{E}_2'(\tau, 1)$ the derivative of a non-holomorphic Eisenstein series $\mathbb{E}_2(\tau, s)$ with respect to $s \in \mathbb{C}$ and a certain modular form $f_\rho(\tau)$.

The above Eisenstein series equals

$$\mathbb{E}_2(\tau, s) := -12\psi(s)E_2(\tau, s),$$

where ψ is a meromorphic function with

$$\psi(s) = -1 + 4(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2})(s-1) + O((s-1)^2)$$

[1]Note this is an arithmetic cycle in the arithmetic Chow group with loglog-growth in the sense of [BKK]

and the expansion of $E_2(\tau,s) = (1/(2\pi i))\partial_\tau E^*(\tau,s)$ (c.f. (2.0.2)) is determined as follows.

Theorem. *The coefficients of the Fourier expansion of the weight* 2 *Eisenstein series for* $\mathrm{SL}_2(\mathbf{Z})$

$$E_2(\tau,1) = \sum_{m\in\mathbf{Z}} a(v,1,m)q^m \tag{1.0.3}$$

and those of the the derivative of $E_2(\tau,s)$ *with respect to* s

$$E_2'(\tau,1) = \sum_{m\in\mathbf{Z}} a'(v,1,m)q^m$$

are given by

$$a(v,1,m) = \begin{cases} \sigma(m) = \sum_{d|m} d & \text{for } m>0 \\ 0 & \text{for } m<0 \end{cases}$$

and by

$$a'(v,1,m) = \begin{cases} \sigma(m)(1/(4\pi mv) + \sigma'(m)/\sigma/m)) & \text{for } m>0 \\ \sigma(m)(\mathrm{Ei}(-4\pi|m|v) + 1/(4\pi|m|v)e^{-4\pi|m|v}) & \text{for } m<0 \end{cases}$$

where with

$$\sigma_s^*(m) := |m|^s \sum_{d|m} d^{-2s}$$

we abbreviate $\sigma'(m)/\sigma(m) := {\sigma^*}'_{1/2}(m)/\sigma^*_{1/2}(m)$.

We also will calculate the constant terms $a(v,1,0)$ and $a'(v,1,0)$ below (c.f. Theorem 2.2) although we won't need them for this work

1.1. Remark. The Fourier expansion $E_2(\tau,s) = \sum a(v,s,m)q^m$ translates via the multiplication by ψ to $\mathbb{E}_2(\tau,s) = \sum A(v,s,m)q^m$, where the first terms of the Taylor expansion at $s=1$ of the Fourier coefficients are given by

$$\begin{aligned} A(v,1,m) &= 12a(v,1,m), \\ A'(v,1,m) &= -48(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2})a(v,1,m) + 12a'(v,1,m). \end{aligned} \tag{1.1.1}$$

The main steps in the proof of our main theorem are to calculate and to compare both sides of (1.0.2) termwise.

Arakelov theory ([BKK] Proposition 7.56) gives us for $m \neq 0$ the relation

$$\widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{Z}_\rho(m) = \operatorname{ht}_{\overline{\mathcal{L}}}(T(m)) + \int_X \widetilde{\Xi}_\rho(v,z,m) c_1(\overline{\mathcal{L}})^2. \tag{1.1.2}$$

Observe that $c_1(\overline{\mathcal{L}})$ is proportional to the hyperbolic measure and, as evaluated later (Remark 5.2), one has for the volume element

$$c_1(\overline{\mathcal{L}})^2 = (18/\pi^2) d\mu(z) = (18/\pi^2) \frac{dx_1 dy_1}{y_1^2} \frac{dx_2 dy_2}{y_2^2}. \tag{1.1.3}$$

Now from Theorem 7.61 in [BKK] p.81 we already know

Proposition. *The Faltings height of $T(m)$ is given by*

$$\operatorname{ht}_{\overline{\mathcal{L}}}(T(m)) = \begin{cases} 0 & \text{if } \; m < 0 \\ 24^2 \big((\sigma(m)((1/2)\zeta(-1) + \zeta'(-1)) & \\ \qquad + \sum_{d|m} (\frac{d \log d}{24} - \frac{\sigma(m) \log m}{48}) \big) & \text{if } \; m > 0. \end{cases} \tag{1.1.4}$$

Therefore we need only to study the integrals

$$\int_X \widetilde{\Xi}_\rho(v,z,m) c_1(\overline{\mathcal{L}})^2 = \int_X \Xi(v,z,m) c_1(\overline{\mathcal{L}})^2 + \int_X \rho(z) \check{\Xi}(v,z,m) c_1(\overline{\mathcal{L}})^2.$$

For the integrals $\int_X \rho(z)\check{\Xi}(v,z,m)c_1(\overline{\mathcal{L}})^2$ we first recall from Proposition 4.3 in Part I that by adding an appropriate zeroth coefficient the q-series

$$\check{\Xi}^+(\tau,z) = \check{\Xi}^+(v,z,0) + \sum_{m \neq 0} \check{\Xi}(v,z,m) q^m$$

is a modular form with respect to $\mathrm{SL}_2(\mathbb{Z})$. Thus, the existence of those integrals implies our first result:

Theorem A. *There exists a non-holomorphic modular form $f_\rho(\tau)$ of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ such that*

$$f_\rho(\tau) = \int_X \rho(z) \check{\Xi}^+(\tau,z) c_1(\overline{\mathcal{L}})^2.$$

1.2. Remark. We observe that the existence of the integral of $\widetilde{\Xi}_\rho$ is guaranteed by Arakelov theory. Then the existence of the integral of $\int_X \rho\check{\Xi}c_1(\overline{\mathcal{L}})^2$ implies the existence of the integral of the Kudla Green function Ξ.

Using $\mathrm{O}(2,2)$-theory we are able to calculate the remaining integrals (see Theorem 4.2).

Theorem B. *We have*

$$\begin{aligned}
&\int_X \Xi(v,z,m))c_1(\overline{\mathcal{L}})^2 \\
(1.2.1) \quad &= \begin{cases} 12\sigma_1(m)(1/(4\pi m v)) & \text{for } m>0 \\ 12\sigma_1(m)((1/(2\pi|m|v))e^{-4\pi|m|v} + \mathrm{Ei}(-4\pi|m|v)) & \text{for } m<0. \end{cases}
\end{aligned}$$

It is now a pleasant exercise to relate these arithmetic intersection numbers for $m \neq 0$ to the Fourier coefficients of the Eisenstein series as in our Main Theorem (see Theorem 2.2). Now, since we already know that the right hand side of our main theorem is a non-holomorphic modular form for $\mathrm{Sl}_2(\mathbb{Z})$ of weight 2, the remaing arithmetic intersection number for $m=0$ must equal the coefficients of the modular form of the right hand side[2]

As already stated in Part I, our treatment of this topic owes a lot to discussions with J.Bruinier, J. Funke and S. Kudla. And this time we even got some local help by hints from H. Brückner and J. Michaliçek. We thank them all.

2 Eisenstein series and its derivatives

We take over classical material from Zagier's article [Za] p.32f. For $\tau = u+iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}\ s > 1$ one has the analytic Eisenstein series

$$\begin{aligned}
E(\tau,s) :=&(1/2){\sum}'_{c,d}\frac{v^s}{|\ c\tau+d\ |^{2s}} \\
=&v^s\zeta(2s) + v^s \sum_{c\in\mathbf{N}}\sum_{d\in\mathbf{Z}} |\ c\tau+d\ |^{-2s}
\end{aligned}$$

[2]We had spent much effort to calculate this identity directly, but we had not been able to do so and would be thankful for any helpful hints.

resp. in normalized version

$$E^*(\tau,s) := \pi^{-s}\Gamma(s)E(\tau,s). \tag{2.0.2}$$

With

$$\zeta^*(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Zagier states the following Fourier development

$$\begin{aligned} E^*(\tau,s) =& v^s\zeta^*(2s) + v^{1-s}\zeta^*(2s-1) \\ &+ 2v^{1/2}\sum_{n\in\mathbb{Z},n\neq 0}\sigma^*_{s-(1/2)}(\mid n\mid)K_{s-(1/2)}(2\pi|n|v)e^{2\pi inu} \end{aligned} \tag{2.0.3}$$

where

$$\sigma^*_\nu(n) :=\mid n\mid^\nu \sum_{d|n} d^{-2\nu} = \sigma^*_{-\nu}(n) \tag{2.0.4}$$

is an entire function in ν and the *K-Bessel function*

$$K_\nu(t) := \int_0^\infty e^{-t\cosh u}\cosh(\nu u)du = K_{-\nu}(t)$$

is entire in ν and exponentially small in t as $t \mapsto \infty$.
We introduce

2.1. Definition.

$$E_2(\tau,s) := (1/(2\pi i))\partial_\tau E^*(\tau,s) = (-1/(4\pi))(\partial_v + i\partial_u)E^*(\tau,s) \tag{2.1.1}$$

and want to study its Taylor expansion at $s=1$. More precisely, we slightly extend the Theorem in the Introduction.

2.2. Theorem. *We have*

$$E_2(\tau,1) = \sum_{m\in\mathbf{Z}} a(v,1,m)q^m \tag{2.2.1}$$

and, denoting by $E_2'(\tau,s)$ the derivative of $E_2(\tau,s)$ with respect to s, we get

$$E_2'(\tau,1) = \sum_{m\in\mathbf{Z}} a'(v,1,m)q^m \tag{2.2.2}$$

with

$$a(v,1,m)=\begin{cases}\sigma(m)=\sum_{d|m}d & \text{for } m>0\\ -1/24+1/(8\pi v) & \text{for } m=0\\ 0 & \text{for } m<0\end{cases}$$

(2.2.3)

$$a'(v,1,m)=\begin{cases}\sigma(m)(1/(4\pi mv)+\sigma'(m)/\sigma(m)) & \text{for } m>0\\ -(1/24)(24\zeta'(-1)+\gamma-1+\log(4\pi v)) & \\ \qquad\qquad -(1/(8\pi v))(-\gamma+\log(4\pi v)) & \text{for } m=0\\ \sigma(m)(\mathrm{Ei}(-4\pi|m|v)+1/(4\pi|m|v)e^{-4\pi|m|v}) & \text{for } m<0.\end{cases}$$

Proof. We have (see for instance Iwaniec [Iw] p.205)

$$\begin{aligned}K_\nu(t):=&\int_0^\infty e^{-t\cosh u}\cosh(\nu u)du\\ =&\frac{\sqrt{\pi}(t/2)^\nu}{\Gamma(\nu+(1/2))}\int_1^\infty e^{-tr}(r^2-1)^{\nu-(1/2)}dr.\end{aligned}$$

Hence, from(2.0.3) we get

$$\begin{aligned}E^*(\tau,s)=&v^s\zeta^*(2s)+v^{1-s}\zeta^*(2s-1)\\ &+\sum_{m\in\mathbf{Z},m\neq0}2\sigma^*_{s-(1/2)}(|m|)\frac{(v\mid m\mid\pi)^s}{\Gamma(s)\sqrt{|m|}}\int_1^\infty e^{-2\pi|m|vr}(r^2-1)^{s-1}dr\,e^{2\pi imu}.\end{aligned}$$

We abbreviate

$$\begin{aligned}c_0(v,s)&:=v^s\zeta^*(2s)+v^{1-s}\zeta^*(2s-1),\\ \check{c}_0(v,s)&:=\partial_v c_0(v,s)=sv^{s-1}\zeta^*(2s)+(1-s)v^{-s}\zeta^*(2s-1),\end{aligned}$$

and, for $m\neq 0$,

$$\begin{aligned}c_m(v,s):=&2\sigma^*_{s-(1/2)}(|m|)\frac{(v|m|\pi)^s}{\Gamma(s)\sqrt{|m|}},\\ I_m(v,s):=&\int_1^\infty e^{-2\pi|m|vr}(r^2-1)^{s-1}dr,\\ J_m(v,s):=&\int_1^\infty e^{-2\pi|m|vr}(r^2-1)^{s-1}rdr,\end{aligned}$$

and get

$$E_2(\tau,s) = -(1/(4\pi))(\check{c}_0(v,s) + \sum_m{}' ((s/v\ -2\pi m)c_m(v,s)I_m(v,s)$$
$$-2\pi|m|c_m(v,s)J_m(v,s))e(mu)$$

i.e.

$$\begin{aligned} E_2(\tau,s) = &- \frac{1}{4\pi}(\check{c}_0(v,s) \\ &+ \sum_{m>0}((s/v)c_m(v,s)I_m(v,s) - 2\pi|m|c_m(v,s)(I_m(v,s) + J_m(v,s))e(mu) \\ &+ \sum_{m<0}((s/v)c_m(v,s)I_m(v,s) + 2\pi|m|c_m(v,s)(I_m(v,s) - J_m(v,s))e(mu)\,). \end{aligned} \tag{2.2.4}$$

Using Maple, one can determine from here the first two terms of the Taylor expansion of each coefficient of $e(mu)$ and hence get the claims in the Theorem. For those who don't like Maple, we give a direct proof in an appendix.

$\square$

2.3. Remark. The sigmas in this calculations are those from the paper by Zagier

$$\sigma_s^*(n) := |n|^s \sum_{d|n,d>0} d^{-2s} = \sigma_{-s}^*(n).$$

Hence one has

$$\sqrt{m}\,\sigma_{1/2}^*(m) = \sum_{d|m} d = \sigma(m).$$

We set

$$\begin{aligned} \sigma'(m)/\sigma(m) &:= \sigma^{*\prime}(m)_{1/2}/\sigma_{1/2}^*(m) \\ &= (\sigma(m)\log m - 2\sum_{d|m} d\log d)/\sigma(m). \end{aligned} \tag{2.3.1}$$

As an immediate consequence to our Theorem, for the coefficients (1.1.1) of the modified Eisenstein series $\mathbb{E}_2'(\tau,1)$, we get

2.4. Corollary. One has

(2.4.1)
$$A'(v,1,m) = -12 \begin{cases} \sigma(m)(4(\zeta'(-1)/\zeta(-1)+1/2) - 1/(4\pi m v) \\ \qquad\qquad +{\sigma^*}'_{1/2}(m)/\sigma^*_{1/2}(m))) & \text{for } m>0 \\ 3\zeta'(-1) - (1/8) + (\gamma/24) + (1/24)\log(4\pi v) \\ \quad +(1/8\pi v)(-48\zeta'(-1) - \gamma + 2 + \log(4\pi v)). & \text{for } m=0 \\ \sigma(|m|)(\mathrm{Ei}(-4\pi|m|v) + 1/(4\pi|m|v)e^{-4\pi|m|v}) & \text{for } m<0 \end{cases}$$

3 Boundary function integral

In Section 2 in Part I we introduced a partition of the unity ρ with respect to the boundary D and the boundary function $\check{\Xi}^+(\tau,z)$

$$\check{\Xi}^+(\tau,z) = \sum_m \check{\Xi}(v,z,m)q^m - (1/2v)t(s+1/s)$$
$$\check{\Xi}(v,z,m) = (1/2)\sum_{-bc=m} \check{\xi}(v,z;b,c)$$

with $t=\sqrt{y_1y_2}$, $s=\sqrt{y_1/y_2}$ (unfortunately we here have the same letter as the one denoting the variable in the zeta and Eisenstein series but the kind reader will know to make the difference) and

$$\check{\xi}(v,z,;b,c) = (t/\sqrt{v})\ (\mathrm{B}(v,s;b,c) - \mathrm{I}(v,s;b,c))$$
$$\mathrm{B}(v,s;b,c) = \int_1^\infty e^{-\pi v(b/s+cs)^2 r} r^{-3/2} dr$$
$$\mathrm{I}(v,s;b,c) = \begin{cases} 4\pi\sqrt{v}\ \min(|\ bs^{-1}\ |,|\ cs\ |)) & \text{if } -bc>0 \\ 0 & \text{if } -bc\le 0. \end{cases}$$

And in Proposition 4.3 of Part I we proved the modularity of $\check{\Xi}^+(\tau,z)$ as a function in τ. From there we come to the following result:

3.1. Theorem. *There exists a modular form f_ρ such that*

(3.1.1)
$$f_\rho(\tau) = \int_X \rho(z)\check{\Xi}^+(\tau,z))d\mu.$$

Proof. As we have modularity in τ of $\check{\Xi}^+$ and since the integral does not affect the $\tau-$variable, the modularity follows as soon as we checked the existence of the integrals. For $m=0$ this is evaluated in the Proposition

3.5 below and for $m \neq 0$ that follows from the three lemmata below as in these we have integrals of type

$$\int_X tF(s)d\mu = \int_X tF(s)dx_1dx_2dsdt/st^3$$

and, as the integrand does not depend on x_1, x_2, once the $s-$integration is done, one has a finite value as

$$\int_{t>t_0} dt/t^2 < \infty.$$

$\square$

3.2. Lemma. *For* $m = -bc < 0$ *and*

$$\tau(m) := \sharp\{d > 0 : d \mid |m|\}$$

one has

$$\sum_{-bc=m} \int_0^\infty B(v,s;b,c)ds/s \leq 2\tau(m)(1/(2\sqrt{|m|v})e^{-4\pi|m|v} + 2\pi\sqrt{|m|v}\,\mathrm{Ei}(-4\pi|m|v))$$

Proof. Replacing s by $s\sqrt{c/b}$ we get

$$\begin{aligned}\int_0^\infty B(v,s;b,c)ds/s &= \int_0^\infty \int_1^\infty e^{-\pi v(b/s+cs)^2 r} dr/r^{3/2}ds/s\\ &= \int_0^\infty \int_1^\infty e^{-\pi v|m|((1/s)^2+s^2)+2)r} dr/r^{3/2}ds/s\end{aligned}$$

and with $s = e^\varphi$ and $\cosh\varphi = 1 + \varphi^2/2 + \dots$ we estimate

$$\begin{aligned}\int_0^\infty B(v,s;b,c)ds/s &= \int_1^\infty \int_{-\infty}^\infty e^{-2\pi vr|m|\cosh 2\varphi} d\varphi e^{-2\pi|m|vr} dr/r^{3/2}\\ &\leq \int_1^\infty \int_{-\infty}^\infty e^{-4\pi v|m|r\varphi^2} d\varphi e^{-4\pi|m|vr} dr/r^{3/2}\\ &= 1/(2\sqrt{|m|v}) \int_1^\infty e^{-4\pi|m|vr} dr/r^2\\ &= 1/(2\sqrt{|m|v})e^{-4\pi|m|v} + 2\pi\sqrt{|m|v}\,\mathrm{Ei}(-4\pi|m|v)).\end{aligned}$$

$\square$

3.3. Lemma. *For $m = -bc > 0$ one has*

$$\sum_{-bc=m} \int_0^\infty B(v,s;b,c)ds/s \le 2\tau(m)(1/(2\sqrt{|m|v}))$$

Proof. Replacing b by $-b$ one has $\sum_{-bc=m} = 2\sum_{b,c>0,bc=m}$ and again s by $s\sqrt{c/b}$ we get this time

$$\begin{aligned}
\int_0^\infty B(v,s;b,c)ds/s &= \int_0^\infty \int_1^\infty e^{-\pi v(b/s+cs)^2 r} dr/r^{3/2} ds/s \\
&= \int_0^\infty \int_1^\infty e^{-\pi v m((1/s)^2+s^2)-2)r} dr/r^{3/2} ds/s
\end{aligned}$$

and with $s = e^\varphi$

$$\begin{aligned}
\int_0^\infty B(v,s;b,c)ds/s &= \int_1^\infty \int_{-\infty}^\infty e^{-2\pi v r m \cosh 2\varphi} d\varphi e^{2\pi m v r} dr/r^{3/2} \\
&\le \int_1^\infty \int_{-\infty}^\infty e^{-4\pi v |m| r \varphi^2} d\varphi dr/r^{3/2} \\
&= 1/(2\sqrt{mv}) \int_1^\infty dr/r^2 \\
&= 1/(2\sqrt{mv}).
\end{aligned}$$

□

3.4. Lemma. *For $m = -bc > 0$ one has*

$$\sum_{-bc=m} \int_0^\infty \min(|b/s|, |cs|)ds/s = 4\tau(m)\sqrt{m}$$

Proof. Replacing s by $s\sqrt{|c|/|b|}$ we get

$$\begin{aligned}
\sum_{-bc=m} \int_0^\infty \min(|b/s|, |cs|)ds/s &= \sum_{-bc=m} \sqrt{m} \int_0^\infty \min(1/s, s)ds/s \\
&= 2\sqrt{m}\tau(m)(\int_0^1 s ds/s + \int_1^\infty ds/s^2) \\
&= 2\sqrt{m}\tau(m) \cdot 2.
\end{aligned}$$

□

3.5. Proposition. *For $m = bc = 0$ one has*

$$\int_X \check{\Xi}^+(v,z,0)d\mu < \infty$$

Proof. From the Remark 2.23 from Part I we get

$$\begin{aligned}2\cdot\check{\Xi}(v,z,0) =&(t/\sqrt{v})(\sum_{b\neq 0} B(v,s;b,0) + \sum_{c\neq 0} B(v,s;0,c) + B(v,s;0,0))\\ =&-2t/\sqrt{v} + t(s+1/s)(1/v+(2/\pi)\zeta(2))\\ &-(2t/\pi)((1/s)\sum_{b\in\mathbb{N}} e^{-\pi s^2b^2/v}/b^2 + s\sum_{c\in\mathbb{N}} e^{-\pi c^2/(s^2v)}/c^2)\end{aligned} \tag{3.5.1}$$

and from (4.2.6) of Part I

$$\begin{aligned}2\cdot\check{\Xi}^+(v,z,0) =&2\cdot\check{\Xi}(v,z,0)-(1/v)t(s+1/s)\\ =&-2t/\sqrt{v}+t(s+1/s)(2/\pi)\zeta(2))\\ &-(2t/\pi)((1/s)\sum_{b\in\mathbb{N}} e^{-\pi s^2b^2/v}/b^2 + s\sum_{c\in\mathbb{N}} e^{-\pi c^2/(s^2v)}/c^2).\end{aligned} \tag{3.5.2}$$

Step 1. We start by integrating

$$\begin{aligned}I' =& \int_{K_1<y_1}\int_{K_2<y_2<T} tdy_1dy_2/(y_2^2y_1^2)\\ =&\int_{K_2}^{T}\int_{K_1}^{\infty} dy_1/y_1^{3/2}dy_2/y_2^{3/2} = (4/\sqrt{K}_1)(1/\sqrt{K}_2-1/\sqrt{T}).\end{aligned}$$

and

$$\begin{aligned}I_0 =& \int_{K_1<y_1}\int_{K_2<y_2<T} y_2dy_1dy_2/(y_2^2y_1^2)\\ =&\int_{K_2}^{T}\int_{K_1}^{\infty} dy_1/y_1^2dy_2/y_2\\ =&\int_{K_2}^{T}(1/K_1)dy_2/dy_2 = (1/K_1)(\log T-\log K_2).\end{aligned}$$

Step 2. For $b\neq 0$ we look at

$$\begin{aligned}I_b :=&(1/b^2)\int_{K_2}^{T} y_2\int_{K_1}^{\infty} e^{-(\pi/v)b^2y_1/y_2}dy_1dy_2/(y_1y_2)^2\\ =&(1/b^2)\int_{K_2}^{T}([e^{-(\pi/v)b^2y_1/y_2}(-1/y_1)]_{K_1}^{\infty}\\ &-\int_{K_1}^{\infty}(\pi b^2/(y_2v))e^{-(\pi/v)b^2y_1/y_2}dy_1/y_1)dy_2/y_2\\ =&(1/b^2)\int_{K_2}^{T}([e^{-(\pi/v)b^2K_1/y_2}(1/K_1)]-\int_{1}^{\infty}(\pi b^2/(y_2v))e^{-(\pi/v)b^2K_1y_1/y_2}dy_1/y_1)dy_2/y_2.\end{aligned}$$

We remind

$$-\mathrm{Ei}\,(-x) = \int_1^\infty e^{-xt}dt/t = -\gamma - \log|x| + x - x^2/(2\cdot 2!) + \dots$$

and have for $T \longrightarrow \infty$ in the first term

$$\begin{aligned}
I_{b,1} &= (1/(b^2K_1))\int_{K_2}^{T} e^{-(\pi/v)b^2K_1/y_2}dy_2/y_2\\
&= (1/(b^2K_1))\int_{1/T}^{1/K_2} e^{-(\pi/v)b^2K_1u}du/u\\
&= (1/(b^2K_1))\int_{1}^{T/K_2} e^{-(\pi/v)b^2K_1u/T}du/u\\
&\leq (1/(b^2K_1))\int_{1}^{\infty} e^{-(\pi/v)b^2K_1u/T}du/u\\
&= (1/(b^2K_1))(-\gamma - \log x + x + \dots)
\end{aligned}$$

where $x = \pi(b^2/v)(K_1/T)$. And for the second term in I_b

$$\begin{aligned}
I_{b,2} :&= (\pi/v)\int_{K_2}^{T}(\int_1^\infty e^{-(\pi/v)b^2K_1y_1/y_2}dy_1/y_1)dy_2/y_2^2\\
&= -(\pi/v)\int_{K_2}^{T}\mathrm{Ei}(-(\pi/v)b^2K_1/y_2)dy_2/y_2^2\\
&= -(\pi/v)\int_{1/T}^{1/K_2}\mathrm{Ei}(-(\pi/v)b^2K_1u)du
\end{aligned}$$

with $\alpha = \pi b^2K_1/v$ we have

$$\begin{aligned}
I_{b,2} =&(\pi/v)\int_{K_2}^{T}\int_1^\infty e^{-\alpha y_1/y_2}dy_1/y_1dy_2/y_2^2\\
=&(\pi/v)\int_1^\infty\int_{1/T}^{1/K_2}[e^{-\alpha y_1u}dudy_1/y_1\\
=&(\pi/(v\alpha))\int_1^\infty[e^{-\alpha y_1/T} - e^{-\alpha y_1/K_2}]dy_1/y_1^2\\
=&(1/(b^2K_1))\int_1^\infty[e^{-\alpha y_1/T} - e^{-\alpha y_1/K_2}]dy_1/y_1^2\\
=&(1/(b^2K_1))([e^{-\alpha/T} - e^{-\alpha/K_2}])\\
&-(\pi/v)\int_1^\infty[e^{-\alpha y_1/T}/T - e^{-\alpha y_1/K_2}/K_2]dy_1/y_1\\
=&(1/(b^2K_1))([e^{-\alpha/T} - e^{-\alpha/K_2}])\\
&-(\pi/(vK_2))\,\mathrm{Ei}(-\pi(b^2/v)K_1/K_2) + (\pi/(vT))\,\mathrm{Ei}(-\pi(b^2/v)K_1/T)
\end{aligned}$$

i.e. something finite for $T \longrightarrow \infty$ as the first terms are harmless and for the last one one has

$$\begin{aligned}(\pi/(vT))\,\mathrm{Ei}(-\pi(b^2/v)K_1/T) =&(\pi/(vT))(-\gamma - \log(-\pi(b^2/v)K_1/T)\\ &+ (-\pi(b^2/v)K_1/T) + \dots .\end{aligned}$$

with $(1/T)\log T \to 0$.

Step 3. We remark that for $T \longrightarrow \infty$ I_0/b^2 and I_b have the same singularity, namely $1/(b^2K_1)\log T$.

Step 4. The same way, we have the the same singularity coming from

$$\begin{aligned}I_0' &= \int_{K_2<y_2}\int_{K_1<y_1<T} y_1 dy_1 dy_2/(y_2^2y_1^2)\\ &= \int_{K_1}^{T}\int_{K_2}^{\infty} dy_2/y_2^2 dy_1/y_1\\ &= \int_{K_1}^{T}(1/K_2)dy_1/dy_1 = (1/K_2)(\log T - \log K_2).\end{aligned}$$

and

$$\begin{aligned}I_c :=&(1/c^2)\int_{K_1}^{T} y_1 \int_{K_2}^{\infty} e^{-(\pi/v)c^2y_2/y_1} dy_1 dy_2/(y_1y_2)^2\\ =&(1/c^2)\int_{K_1}^{T}([e^{-(\pi/v)c^2y_2/y_1}(-1/y_1)]_{K_2}^{\infty}\\ &- \int_{K_2}^{\infty}(\pi c^2/(y_1v))e^{-(\pi/v)c^2y_2/y_1}dy_2/y_2)dy_1/y_1\\ =&(1/c^2)\int_{K_1}^{T}([e^{-(\pi/v)c^2K_2/y_1}(1/K_2)]\\ &- \int_{1}^{\infty}(\pi c^2/(y_1v))e^{-(\pi/v)c^2K_2y_2/y_1}dy_2/y_2)dy_1/y_1.\end{aligned}$$

Hence all together adds up to something finite. $\square$

4 Kudla's Green function integral for $m \neq 0$

At first we remark that, as explained at the end of the Introduction, it follows immediately from Theorem 3.1 that for $m \neq 0$ the integrals in question

exist.

We look at the Green function integral

4.1. Definition.

$$
\begin{aligned}
I_m &:= \int_{\Gamma\backslash\mathfrak{H}\times\Gamma\backslash\mathfrak{H}} \Xi(v,z,m)d\mu(z) \\
&= \int_{\Gamma\backslash\mathfrak{H}\times\Gamma\backslash\mathfrak{H}} (1/2)\sum_{M\in L_m}\xi(v,z,m)d\mu(z) \qquad (4.1.1)
\end{aligned}
$$

and want to prove the following

4.2. Theorem. *With $\sigma(m) := \sum_{d|m} d$ one has*

(4.2.1)

$$
I_m = \begin{cases} \sigma(|m|)(\pi/3)(1/(2v|m|))(e^{-4\pi v|m|} + 4\pi v|m|\mathrm{Ei}(-4\pi|m|v)) & \text{for } m<0 \\ \sigma(m)(\pi/3)(1/(2vm)) & \text{for } m>0. \end{cases}
$$

Proof. At first we assemble some tools. For $m\in\mathbb{N}$ and

$$
\begin{aligned}
\Gamma &= \mathrm{SL}(2,\mathbb{Z}), \\
L_m &= \{M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_2(\mathbb{Z});\ \det M = m\}, \\
L_m^* &= \{M\in L_m;\ M \text{ primitive}\},
\end{aligned}
$$

one has the standard facts (see for instance Ogg's book [Og] p.II-7 and IV-4)

$$
\begin{aligned}
L_m &= \cup_{ad=m,d>0,a|d}\,\Gamma\begin{pmatrix} a & \\ & d\end{pmatrix}\Gamma \\
&= \cup_{ad=m,d>0,\,b \mod d}\,\Gamma\begin{pmatrix} a & b \\ & d\end{pmatrix} \qquad (4.2.2)
\end{aligned}
$$

and

$$
L_m^* = \Gamma\begin{pmatrix} m & \\ & 1\end{pmatrix}\Gamma = \sqcup_\alpha \Gamma\alpha \qquad (4.2.3)
$$

with

$$
\alpha = \begin{pmatrix} a & b \\ & d\end{pmatrix},\ ad=m, d>0, 0\le b<d, (a,b,d)=1.
$$

One has

$$
[L_m^*:\Gamma] = m\prod_{p|m}(1+(1/p)) =: \psi(m)
$$

and also

$$[\Gamma : \Gamma_0(m)] = \psi(m).$$

Moreover, we have

$$[L_m : \Gamma] = \sum_{d|m} d = \sigma(m)$$

and hence the formula (which is easily verified using the multiplicativity of σ, see for instance Rankin [Ra] p. 285)

$$\sigma(m) = \sum_{n^2|m} \psi(m/n^2). \tag{4.2.4}$$

Γ acts transitively by right multiplication on $\Gamma\backslash L_m^*$ with isotropy group $\Gamma_0(m)$ at the coset $\Gamma\begin{pmatrix} m & \\ & 1\end{pmatrix}$ (for example as in Knapp [Kn] p.256, Proposition 9.3). Hence, one has a bijection

$$L_m^* \simeq \Gamma\begin{pmatrix} m & \\ & 1\end{pmatrix}(\Gamma_0(m)\setminus\Gamma). \tag{4.2.5}$$

It also is a standard fact that one has

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \int_{\Gamma\backslash\mathbb{H}} \frac{dxdy}{y^2} = \pi/3. \tag{4.2.6}$$

and for $m > 0$ (e.g. [FB] p.375)

$$\mathrm{vol}(\Gamma_0(m)\backslash\mathbb{H}) = \psi(m)\pi/3. \tag{4.2.7}$$

After the preparation of these tools, we come to calculate the Green function integral

$$I_m := \int_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} \Xi(v, z, m)d\mu(z)$$

with

$$d\mu(z) = d\mu(z_1)d\mu(z_2) = \prod_{j=1,2} \frac{dx_j dy_j}{y_j^2}.$$

We do this in several steps.

Step 1. The integral for squarefree positive m

To simplify things, we start by treating the special case of squarefree $m > 0$ where $L_m^* = L_m$.

As $\overline{\Gamma} = \Gamma \times \Gamma$ acts on L_m via $M \mapsto \gamma_1 M^t\gamma_2 =: M^\gamma$, we have

$$\begin{aligned} I_m &= \int_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} (1/2) \sum_{M\in L_m^*} \xi(v,z,M)d\mu(z) \\ &= \int_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} (1/2) \sum_{\gamma_1\in\Gamma,\beta\in(\Gamma_0(m)\backslash\Gamma)} \xi(v,(z_1,z_2);\gamma_1 \begin{pmatrix} m & \\ & 1\end{pmatrix} {}^t\beta)d\mu(z) \\ &= \int_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} (1/2) \sum_{\gamma_1\in\Gamma,\beta\in(\Gamma_0(m)\backslash\Gamma)} \xi(v,(\gamma_1^{-1}z_1,\beta^{-1}z_2); \begin{pmatrix} m & \\ & 1\end{pmatrix})d\mu(z) \end{aligned}$$

where we use the homogenity $\xi(gz, M^g) = \xi(z, M)$. Hence, one has

$$I_m = (1/2)\int_{\mathbb{H}\times(\Gamma_0(m)\backslash\mathbb{H})} \xi(v,z,\begin{pmatrix} m & \\ & 1\end{pmatrix})d\mu(z)$$

with

$$\xi(v,z,\begin{pmatrix} m & \\ & 1\end{pmatrix}) = \int_1^\infty e^{-2\pi v R(z,\left(\begin{smallmatrix} m & \\ & 1\end{smallmatrix}\right))u} du/u$$

and

$$R(z,\begin{pmatrix} m & \\ & 1\end{pmatrix} = (1/(2y_1y_2))|m + z_1z_2|^2.$$

We simplify this by changing two times our coordinates.

i) The change

$$(z_1,z_2) \longmapsto (z_1, mz_2)$$

leads to $d\mu(z) \mapsto d\mu(z)$ and

$$R(z,\left(\begin{smallmatrix} m & \\ & 1\end{smallmatrix}\right)) \longmapsto mR(z,\left(\begin{smallmatrix} 1 & \\ & 1\end{smallmatrix}\right)).$$

ii) For g_{z_2} with $g_{z_2}(i) = z_2$ we take

$$z = (z_1,z_2) \longmapsto g(z) = ({}^tg_{z_2}(z_1), g_{z_2}^{-1}(z_2)) =: ({z_1}', i)$$

and have

$$\begin{aligned} R(z,\left(\begin{smallmatrix} 1 & \\ & 1\end{smallmatrix}\right)) =& R(g(z),\left(\begin{smallmatrix} 1 & \\ & 1\end{smallmatrix}\right)^g) \\ =& R({z_1}', i;, \left(\begin{smallmatrix} 1 & \\ & 1\end{smallmatrix}\right)) \\ =& (1/(2{y_1}')) \mid 1 + i{z_1}' \mid^2 \end{aligned}$$

and finally

$$\begin{aligned} I_m =& \ \mathrm{vol}(\Gamma_0(m)\setminus\mathbb{H})(1/2)\textstyle\int_{\mathbb{H}}(\int_1^\infty e^{-2\pi vmR({z_1}',i;\left(\begin{smallmatrix} 1 & \\ & 1\end{smallmatrix}\right))u} du/u)d\mu({z_1}') \\ =& \ \mathrm{vol}(\Gamma_0(m)\setminus\mathbb{H})(1/2)\textstyle\int_{\mathbb{H}}(\int_1^\infty e^{-\pi vm(1/(y_1))((1-y_1)^2+x_1^2)u} du/u)d\mu(z_1). \end{aligned}$$

Thus one is reduced to a two-dimensional integral

$$(4.2.8) \qquad I'_m = \int_{\mathbb{H}} \int_1^\infty e^{-\pi v m(1/(y_1))((1-y_1)^2+x_1^2)u} du/u) d\mu(z_1).$$

Using parts of the $\mathrm{SO}(1,2)-$theory as for instance in Brunier-Funke [BF], we change coordinates

$$\mathbb{R}^2 \longrightarrow \mathbb{H},\ (r,\varphi) \longmapsto (z = x+iy)$$

with

$$y = 1/(\cosh r - \sinh r \cos\varphi),\ x = -\sinh r \sin\varphi/(\cosh r - \sinh r\cos\varphi).$$

i.e.,

$$\begin{aligned} (x^2+y^2+1)/(2y) &= \cosh r \\ (x^2+y^2-1)/(2y) &= \sinh r \cos\varphi \\ -x/y &= \sinh r \sin\varphi. \end{aligned}$$

A small calculation shows that one has

$$d\mu(z) = dx \wedge dy/y^2 = \sinh r\ dr \wedge d\varphi.$$

We get

$$\begin{aligned} I'_m &= \int_{\mathbb{H}} (\int_1^\infty e^{-\pi v m(1/y)((1+x^2+y^2-2y)u} du/u) d\mu(z) \\ &= \int_0^{2\pi} \int_0^\infty (\int_1^\infty e^{-2\pi v m(\cosh r-1)u} du/u) \sinh r dr d\varphi \\ &= 2\pi \int_1^\infty \int_1^\infty e^{-2\pi v m u(t-1)} dt du/u \\ &= 2\pi \int_1^\infty (\int_1^\infty e^{-2\pi v m u t} dt) e^{\pi v m u} du/u \\ &= 2\pi \int_1^\infty [e^{-2\pi v m u t}/(-2\pi v m u)]_1^\infty\, e^{2\pi v m u} du/u \\ &= (1/vm) \int_1^\infty u^{-2} du \\ &= (1/vm). \end{aligned}$$

and hence

$$(4.2.9) \qquad I_m = \mathrm{vol}(\Gamma_0(m)\setminus\mathbb{H})(1/2)(1/vm) = \sigma(m)\pi/(6vm).$$

Step 2. The integral for squarefree negative m

For negative m we need slight changes in the calculation of the integral I_m in the second part of our proof. At first one has

$$L_m^* \simeq \Gamma \begin{pmatrix} m & \\ & 1 \end{pmatrix} (\Gamma_0(|m|) \setminus \Gamma).$$

Then the transformation

$$(z_1, z_2) \longmapsto (z_1, mz_2)$$

transforms $\mathbb{H} \times \mathbb{H}$ to $\mathbb{H} \times \overline{\mathbb{H}}$. Hence, in the next step, we have to replace the old g_{z_2} by another one with $g_{z_2}(-i) = z_2$ and come to

$$\begin{aligned}
I_m =& \mathrm{vol}(\Gamma_0(|m|) \setminus \mathbb{H})(1/2) \int_{\mathbb{H}} (\int_1^\infty e^{-2\pi v|m|R(z_1{}', -i; (\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}))u)} du/u) d\mu(z_1{}') \\
=& \mathrm{vol}(\Gamma_0(|m|) \setminus \mathbb{H})\,(1/2) \int_{\mathbb{H}} (\int_1^\infty e^{-\pi(1/y_1)((1+y_1)^2 + x_1^2)v|m|u} du/u) d\mu(z_1) \\
=& \mathrm{vol}(\Gamma_0(|m|) \setminus \mathbb{H})\,(1/2) \int_{\mathbb{H}} (\int_1^\infty e^{-\pi(1/y_1)((1+x_1^2+y_1^2+2y_1)v|m|u} du/u) d\mu(z_1) \\
=& \mathrm{vol}(\Gamma_0(|m|) \setminus \mathbb{H})\,(1/2) 2\pi \int_0^\infty (\int_1^\infty e^{-2\pi v|m|(\cosh r+1)u} du/u) \sinh r dr \\
=& \mathrm{vol}(\Gamma_0(|m|) \setminus \mathbb{H})\,(1/2) 2\pi \int_1^\infty (\int_1^\infty e^{-2\pi v|m|(t+1)u} dt) du/u \\
=& \mathrm{vol}(\Gamma_0(|m|) \setminus \mathbb{H})\,(1/2)(-1/vm) \int_1^\infty e^{-4\pi v|m|u} u^{-2} du
\end{aligned}$$

(4.2.10)

$$=\mathrm{vol}(\Gamma_0(|m|) \setminus \mathbb{H})\,(1/2)((-1/vm)e^{-4\pi v|m|} - 4\pi \int_1^\infty e^{-4\pi v|m|u} du/u).$$

Step 3. The integral for general $m \neq 0$

We use the results which we already have and for positive m calculate

$$\begin{aligned}
I_m &= \int_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} (1/2) \sum_{M \in L_m} \xi(z, M) d\mu(z) \\
&= \int_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} \sum_{n^2|m} (1/2) \sum_{M \in L^*_{m/n^2}} \xi(z, nM) d\mu(z) \\
&= \sum_{n^2|m} \int_{\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}} (1/2) \sum_{M \in L^*_{m/n^2}} \xi(z, nM) d\mu(z).
\end{aligned}$$

With

$$L^*_{m/n^2} = \Gamma \begin{pmatrix} m/n^2 & \\ & 1 \end{pmatrix} (\Gamma_0(m/n^2) \setminus \Gamma)$$

we get

$$I_m = \quad (1/2) \sum_{n^2|m} \int_{\mathbb{H}} \int_{\Gamma_0(m)\setminus\mathbb{H}} \xi(z_1, z_2; n \begin{pmatrix} m/n^2 & \\ & 1 \end{pmatrix}) d\mu(z).$$

One has

$$R(z, n \begin{pmatrix} m/n^2 & \\ & 1 \end{pmatrix}) = (1/2y_1y_2) \mid m/n + nz_1z_2 \mid^2$$

and changing the coordinates, as in the second part of our proof above, this time by $(z_1, z_2) \longmapsto (z_1, mz_2/n^2)$ we get

$$R(z, n \begin{pmatrix} m/n^2 & \\ & 1 \end{pmatrix}) \longmapsto (1/2y_1y_2)m \mid 1 + z_1z_2 \mid^2 = mR(z, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})$$

and hence, as in the second part above, via $z \longmapsto g(z) = (z_1, i)$

$$\begin{aligned} I_m = &\quad (1/2) \textstyle\sum_{n^2|m} \mathrm{vol}(\Gamma_0(|m/n^2|) \setminus \mathbb{H}) \int_{\mathbb{H}} \int_1^\infty e^{-2\pi m R(z_1, i; \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right))u} (du/u) d\mu(z_1) \\ = &\quad (1/2) \textstyle\sum_{n^2|m} \mathrm{vol}(\Gamma_0(|m/n^2|) \setminus \mathbb{H}) I'_m \\ = &\quad (1/2) \textstyle\sum_{n^2|m} \psi(m/n^2) I'_m = \sigma_1(m) \kappa \, I'_m \\ = &\quad \sigma_1(m)(\pi/3)(1/2)(1/vm) \end{aligned}$$

where we used the formulae (4.2.2) and (4.2.4) from the first part of the proof and put $\kappa := \mathrm{vol}(\Gamma \setminus \mathbb{H}) = \pi/3$.

For negative m one gets the same way, with m replaced by $|m|$, analogously the formula (4.2.10) from above

$$I_m = \sigma_1(|m|)(\pi/3)(1/v|m|)(e^{-4\pi v|m|} + 4\pi v|m| \mathrm{Ei}(-4\pi|m|v)).$$

$\square$

5 Proof of the Main Theorem

Now we relate the results obtained for $m \neq 0$ in Theorem 4.2 for the Green function integral I_m to the Fourier coefficients of our modified Eisenstein series in Corollary 2.4.

5.1. Theorem. *For $m \neq 0$, with* (1.1.2) *one has*

$$\widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{Z}(m) = \mathrm{ht}_{\overline{\mathcal{L}}}(T(m)) + \int_X \widetilde{\Xi}(v,z,m))c_1(\bar{\mathcal{L}})^2$$
$$= A'(v,1,m). \tag{5.1.1}$$

Proof. We already observed

$$A'(v,1,m) = -48(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2})a(v,1,m) + 12a'(v,1,m).$$

Hence, for $m > 0$, from Theorem 2.2 resp. Corollary 2.4 and the Remark 2.3 on the different sigmas one has

$$\begin{aligned} A'(v,1,m) &= -48\sigma(m)(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}) \\ &\quad + 12\sigma(m)(1/(4\pi mv) + \sigma^{*\prime}(m)_{1/2}/\sigma^*_{1/2}(m)) \\ &= -48\sigma(m)(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}) \\ &\quad + 12(\sigma(m)/(4\pi mv) + (\sigma(m)\log m - 2\sum_{d|m} d\log d)). \end{aligned}$$

On the other hand, as already remarked in the introduction, from Theorem 7.62 in [BKK] and Theorem 4.2

$$\begin{aligned} \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{Z}(m) &= \mathrm{ht}_{\overline{\mathcal{L}}}(T(m)) + c\int_X \widetilde{\Xi}(v,z,m))d\mu \\ &= 24^2(\sigma(m)((1/2)\zeta(-1) + \zeta'(-1)) + \sum_{d|m}(\frac{d\log d}{24} - \frac{\sigma(m)\log m}{48})) \\ &\quad + c\sigma_1(m)(\pi/3)2\pi(1/(4\pi mv)). \end{aligned}$$

For $m < 0$ one has by Theorem 2.2 $a(v,1,m) = 0$ and

$$A'(v,1,m) = -(\mathrm{Ei}(-4\pi|m|v) + e^{-4\pi|m|v}/(4\pi|m|v))\sigma_1(m))$$

and again from Theorem 4.2

$$\begin{aligned} \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{Z}(m) &= \mathrm{ht}_{\overline{\mathcal{L}}}(T(m)) + c\int_X \widetilde{\Xi}(v,z,m))d\mu \\ &= c\sigma_1(m)\pi^2(2/3)(\mathrm{Ei}(-4\pi|m|v) + e^{-4\pi|m|v}/(4\pi|m|v)). \end{aligned}$$

In both cases we get the equality we claimed with $c = 18/\pi^2$. $\square$

5.2. Remark. The constant c is explained by the fact that in the context of [BKK] and [BBK] one has

$$\int c_1(\overline{\mathcal{L}})^2 = c_1(\overline{\mathcal{L}}) \cdot c_1(\overline{\mathcal{L}}) = T(1) \cdot T(1) = 2$$

with

$$c_1(\overline{\mathcal{L}}) = 12\,(dx_1dy_1/(4\pi y_1^2) + dx_2dy_2/(4\pi y_2^2))$$

and

$$c_1(\overline{\mathcal{L}})^2 = 2 \cdot (12/4\pi)^2(dx_1dy_1dx_2dy_2/(y_1y_2)^2) = (18/\pi^2)d\mu(z).$$

Finally, we have all the material for the

Proof of the Main Theorem. As we know the modularity of both sides of (1.0.2) the equality for $m \neq 0$ following from Theorem 5.1 is sufficient (and, hence, gives the value for $m = 0$ we up to now did not determine directly).$\square$

6 Remarks towards a direct calculation of the constant term

Though we don't really need this, to strive for some completeness, we will make some remarks concerning the case $m = 0$.

For $m = 0$, as a consequence of the log-log-singularity of the metric on $\overline{\mathcal{L}}$, $\mathrm{ht}_{\overline{\mathcal{L}}}(T(0))$ is not defined and by the same reason $\int_X \widetilde{\Xi}_\rho(v, z, 0)d\mu$ does not exist. Therefore instead of (1.1.2), we have to use the formula

$$\begin{aligned} \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{Z}_\rho(0) = &\int_X (\widetilde{\Xi}_\rho(v, z, 0) + (1/24 - 1/(8\pi v))g_0)d\mu \\ &+ \widetilde{c}(\zeta'(-1)/\zeta(-1) + 1/2), \end{aligned} \tag{6.0.1}$$

where

$$g_0 := -\log \parallel \Delta(z_1)\Delta(z_2) \parallel^2 . \tag{6.0.2}$$

This formula comes out as here one has

$$\widehat{Z}_\rho(0) = (T(0), \breve{\widetilde{\Xi}}_\rho)$$
$$\widehat{T}(0) = (T(0), c_0 g_0),\ \ c_0 := -1/24 + 1/8\pi v$$

and

$$\begin{aligned} \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{Z}_\rho(0) &= \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot (\widehat{T}(0) + (\widehat{Z}_\rho(0) - \widehat{T}(0)) \\ &= \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{T}(0) + \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot (\widehat{Z}_\rho(0) - (\widehat{T}(0)) \end{aligned} \tag{6.0.3}$$

where the first summand is known to be ([BKK] Theorem 7.61)

$$\begin{aligned} \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot \widehat{T}(0) &= \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot c_0 \widehat{c}_1(\overline{\mathcal{L}}) \\ &= \widetilde{c}(\zeta'(-1)/\zeta(-1) + 1/2),\ \widetilde{c} = -(1/2)12^2 c_0 \end{aligned}$$

and (using the formulae (3.3.1), (3.1.1), and (3.0.5) from Part I) the last one gives the integral above in the formula (6.0.1)

$$\begin{aligned} \widehat{c}_1(\overline{\mathcal{L}})^2 \cdot (\widehat{Z}_\rho(0) - (\widehat{T}(0)) &= \int_X (\breve{\widetilde{\Xi}}_\rho(v,z,0) - c_0 g_0) c_1(\overline{\mathcal{L}})^2 \\ &= \int_X (\Xi(v,z,0) - c_0 g_0) c_1(\overline{\mathcal{L}})^2 + \int_X \rho(z) \breve{\Xi}^+(v,z,0) c_1(\overline{\mathcal{L}})^2 \end{aligned} \tag{6.0.4}$$

We already fixed

$$f_\rho(0) = \int_X \rho(z) \breve{\Xi}^+(v,z,0)) d\mu.$$

Thus, if one wants to avoid the reasoning from the proof above, for a direct proof of the $m = 0-$case it remains to show that $(\pi^2/18)A'(v,z,0)$ has the same value as

$$\begin{aligned} &\int_X (\Xi(v,z,0) + (1/24 - 1/(8\pi v)) g_0) d\mu \\ &\qquad = \int_X ((\Xi^*(v,z,0) - (1/24) g_0) + (1/24 - 1/(8\pi v)) g_0) d\mu \\ &\qquad = \int_X (\Xi^*(v,z,0) - 1/(8\pi v) g_0) d\mu. \end{aligned} \tag{6.0.5}$$

Observe, if we split the integrals, then we would get two divergent integrals where for the integral over g_0 the relevant terms in an asymptotic expansion at the boundary had been calculated already in [K].

A Appendix: Fourier expansion of $E_2(\tau, s)$

Now, here we add the direct proof of Theorem 2.2 where in the main part we relied on Maple calculations.

A.1. Theorem. *We have*

(A.1.1)
$$E_2(\tau,1) = \sum_{m\in\mathbf{Z}} a(v,1,m)q^m$$

and, denoting by $E_2'(\tau,s)$ the derivative of $E_2(\tau,s)$ with respect to s, we get

(A.1.2)
$$E_2'(\tau,1) = \sum_{m\in\mathbf{Z}} a'(v,1,m)q^m$$

with

$$a(v,1,m) = \begin{cases} \sigma_1(m) & \text{for } m>0 \\ -1/24 + 1/(8\pi v) & \text{for } m=0 \\ 0 & \text{for } m<0 \end{cases}$$

(A.1.3)
$$a'(v,1,m) = \begin{cases} \sigma_1(m)(1/(4\pi m v) + \sigma'/\sigma) & \text{for } m>0 \\ -(1/24)(24\zeta'(-1)+\gamma-1+\log(4\pi v)) & \\ \quad -(1/(8\pi v))(-\gamma+\log(4\pi v)) & \text{for } m=0 \\ \sigma_1(m)(\mathrm{Ei}(-4\pi|m|v)+1/(4\pi|m|v)e^{-4\pi|m|v}) & \text{for } m<0 \end{cases}$$

where with σ^* as in (2.0.4)

(A.1.4)
$$\sigma := \sigma^*_{1/2}(m),\ \ \sigma' := {\sigma^*}'_{1/2}(m).$$

Proof. 1. We have (see for instance Iwaniec [Iw] p.205)

$$\begin{aligned} K_\nu(t) :=& \int_0^\infty e^{-t\cosh u}\cosh(\nu u)du \\ =&\frac{\sqrt{\pi}(t/2)^\nu}{\Gamma(\nu+(1/2))}\int_1^\infty e^{-tr}(r^2-1)^{\nu-(1/2)}dr. \end{aligned}$$

Hence, from(2.0.3) we get

$$\begin{aligned} E^*(\tau,s) =& v^s\zeta^*(2s) + v^{1-s}\zeta^*(2s-1) \\ & + \sum_{m\in\mathbf{Z}, m\neq 0} 2\sigma^*_{s-(1/2)}(|m|)\frac{(v\mid m\mid\pi)^s}{\Gamma(s)\sqrt{|m|}}\int_1^\infty e^{-2\pi|m|vr}(r^2-1)^{s-1}dr\, e^{2\pi imu}. \end{aligned}$$

As in the rudimentary proof of Theorem 2.2 we abbreviate

$$\begin{aligned} c_0(v,s) &:= v^s\zeta^*(2s) + v^{1-s}\zeta^*(2s-1), \\ \check{c}_0(v,s) &:= \partial_v c_0(v,s) = sv^{s-1}\zeta^*(2s) + (1-s)v^{-s}\zeta^*(2s-1), \end{aligned}$$

and, for $m \neq 0$,

$$\begin{aligned} \alpha &:= 2\pi|m|v, \\ c_m(v,s) &:= 2\sigma^*_{s-(1/2)}(|m|)\frac{(v|m|\pi)^s}{\Gamma(s)\sqrt{|m|}}, \\ I_m(v,s) &:= \int_1^\infty e^{-2\pi|m|vr}(r^2-1)^{s-1}dr, \\ J_m(v,s) &:= \int_1^\infty e^{-2\pi|m|vr}(r^2-1)^{s-1}rdr, \end{aligned}$$

and get

$$\begin{aligned} E_2(\tau,s) = -(1/(4\pi))(\check{c}_0(v,s) + {\sum_m}'((s/v\ -2\pi m)c_m(v,s)I_m(v,s) \\ -2\pi|m|c_m(v,s)J_m(v,s))e(mu) \end{aligned}$$

i.e.

$$\begin{aligned} E_2(\tau,s) = &-\frac{1}{4\pi}(\check{c}_0(v,s) \\ &+\sum_{m>0}((s/v)c_m(v,s)I_m(v,s) - 2\pi|m|c_m(v,s)(I_m(v,s)+J_m(v,s))e(mu) \\ &+\sum_{m<0}((s/v)c_m(v,s)I_m(v,s) + 2\pi|m|c_m(v,s)(I_m(v,s)-J_m(v,s))e(mu)\ ). \end{aligned} \tag{A.1.5}$$

2. Denoting by ' the derivation with respect to s, one has

$$\begin{aligned} E_2'(\tau,s) = -\frac{1}{4\pi}(\check{c}_0'(v,s) \\ + \sum_{m>0}(((1/v)c_m\ &+(s/v)c_m')I_m - 2\pi|m|c_m'(I_m+J_m) \\ &+(s/v)c_mI_m' - 2\pi|m|c_m(I_m'+J_m'))e(mu) \\ + \sum_{m<0}(((1/v)c_m\ &+(s/v)c_m')I_m + 2\pi|m|c_m'(I_m-J_m) \\ &+(s/v)c_mI_m' + 2\pi|m|c_m(I_m'-J_m'))e(mu)\ ). \end{aligned} \tag{A.1.6}$$

From

$$c_m(v,s) := 2\sigma^*_{s-(1/2)}(\mid m \mid)\frac{(v \mid m \mid \pi)^s}{\Gamma(s)\sqrt{|m|}}$$

we get

$$c_m(v,1) = \sigma^*_{1/2}(\mid m \mid)\alpha/\sqrt{|m|}$$

and with $\sigma_s := \sigma^*_s(|m|)$

$$c'_m(v,s) = ((\sigma'_{s-(1/2)}/\sigma_{s-1/2}) + \log(\alpha/2) - (\Gamma'(s)/\Gamma(s)))c_m(v,s).$$

Using $\Gamma'(1) = -\gamma, \gamma$ the Euler constant, we have

$$c'_m(v,1) = ((\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2) + \gamma)c_m(v,1).$$

Now, we can write for $s = 1$

$$\begin{aligned} E'_2(\tau,1) = \quad & -(1/(4\pi))(\check{c}'_0(v,1) \\ & + \textstyle\sum_{m>0}((1/v)c_m((1+ \quad (\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2) + \gamma)I_m \\ & \qquad - \alpha((\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2) + \gamma)(I_m + J_m) \\ & \qquad + I'_m - \alpha(I'_m + J'_m))e(mu) \\ & + \textstyle\sum_{m<0}((1/v)c_m((1+ \quad (\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2) + \gamma)I_m \\ & \qquad + \alpha((\sigma'_{1/2}/\sigma_{1/2}) + \log(\alpha/2) + \gamma)(I_m - J_m) \\ & \qquad + I'_m + \alpha(I'_m - J'_m))e(mu)\,). \end{aligned}$$

One has to determine the values at $s = 1$ of the functions in this relation: From the definitions one has

$$\begin{aligned} I_m(v,1) &= e^{-\alpha}(1/\alpha), \\ J_m(v,1) &= e^{-\alpha}((1/\alpha) + (1/\alpha^2)), \end{aligned}$$

i.e.

$$\begin{aligned} I_m(v,1) - J_m(v,1) &= -e^{-\alpha}(1/\alpha^2), \\ I_m(v,1) + J_m(v,1) &= e^{-\alpha}((2/\alpha) + (1/\alpha^2)), \end{aligned}$$

and, hence, via (A.1.5) immediately the formulae for the $a(v,1,m), m \neq 0$, in the theorem.

3. For the other terms we will use the well known relation

$$\Gamma'(1) = \int_0^\infty e^{-t}\log t\,dt = -\gamma$$

and its consequence

$$\int_0^\infty e^{-\alpha t}\log t\,dt = -(1/\alpha)(\gamma + \log\alpha).$$

Moreover, one has using partial integration

$$\int_0^\infty e^{-\alpha t} t \log t\, dt = \quad (1/\alpha)((1/\alpha) - (1/\alpha)(\gamma + \log\alpha)),$$

and with $-\mathrm{Ei}(-s) = \int_1^\infty e^{-st} dt/t$

$$\begin{aligned} \int_2^\infty e^{-\alpha t} \log t\, dt = &\quad (1/\alpha)(-\mathrm{Ei}(-2\alpha) + e^{-2\alpha} \log 2) \\ \int_2^\infty e^{-\alpha t} t \log t\, dt = &\quad (1/\alpha^2)(-\mathrm{Ei}(-2\alpha) + e^{-2\alpha}(2\alpha \log 2 + 1 + \log 2)). \end{aligned}$$

Hence we get

$$\begin{aligned} I_m'(v,s) = &\quad \int_1^\infty e^{-\alpha r}(r^2-1)^{s-1} \log(r^2-1)\, dr \\ I_m'(v,1) = &\quad \int_1^\infty e^{-\alpha r} \log(r-1)\, dr + \int_1^\infty e^{-\alpha r} \log(r+1)\, dr \\ = &\quad e^{-\alpha} \int_0^\infty e^{-\alpha r} \log r\, dr + e^{\alpha} \int_2^\infty e^{-\alpha r} \log r\, dr \\ = &\quad e^{\alpha}(1/\alpha)(-\mathrm{Ei}(-2\alpha)) - e^{-\alpha}(1/\alpha)(\log(\alpha/2) + \gamma) \end{aligned}$$

and similarly

$$\begin{aligned} J_m'(v,s) = &\quad \int_1^\infty e^{-\alpha r}(r^2-1)^{s-1} r \log(r^2-1)\, dr \\ J_m'(v,1) = &\quad \int_1^\infty e^{-\alpha r} r \log(r-1)\, dr + \int_1^\infty e^{-\alpha r} r \log(r+1)\, dr \\ = &\quad e^{-\alpha} \int_0^\infty e^{-\alpha r}(r+1) \log r\, dr + e^{\alpha} \int_2^\infty e^{-\alpha r}(r-1) \log r\, dr \\ = &\quad e^{\alpha}(\mathrm{Ei}(-2\alpha))((1/\alpha) - (1/\alpha^2)) + e^{-\alpha}(1/\alpha^2)(2 - (\log(\alpha/2) + \gamma)(1+\alpha)). \end{aligned}$$

One has

$$\begin{aligned} I_m'(v,1) + J_m'(v,1) = &\quad e^{\alpha}(-\mathrm{Ei}(-2\alpha)(1/\alpha^2)) \\ &\quad + e^{-\alpha}(1/\alpha^2)(2 - (\log(\alpha/2) + \gamma) - 2\alpha(\log(\alpha/2) + \gamma)) \\ I_m'(v,1) - J_m'(v,1) = &\quad e^{\alpha}(\mathrm{Ei}(-(2\alpha))((1/\alpha^2) - (2/\alpha)) \\ &\quad - e^{-\alpha}(1/\alpha^2)(2 - (\log(\alpha/2) + \gamma)). \end{aligned}$$

Hence, from (A.1.6), in the equation for $E_2'(\tau,1)$ the coefficient of $c_m(v,1)/v$ for $m > 0$ comes out as

$$e^{-\alpha}(-2\sigma'/\sigma - (1/\alpha))$$

and for $m < 0$ as

$$e^{-\alpha}(-(1/\alpha)) - e^{\alpha} 2\mathrm{Ei}(-(2\alpha)).$$

Remembering that for $m > 0$ with $q = e(u + iv)$ one has $q^m = e^{-\alpha}e(mu)$ and for $m < 0$ $q^m = e^{\alpha}e(mu)$, we get with $\sigma := \sigma^*_{1/2}(|m|)$

$$\begin{aligned} E_2'(\tau,1) = \ & -(1/4\pi)\check{c}_0'(v,1) \\ & + \textstyle\sum_{m>0}(\sqrt{|m|}/(4\pi))(\sigma/(4\pi mv) + \sigma')q^m \\ & + \textstyle\sum_{m<0}(\sqrt{|m|}/(2\pi))\sigma(-\mathrm{Ei}(-(2\alpha)) + e^{4\pi mv}/(4\pi mv))q^m. \end{aligned}$$

One can simplify this a bit using the relation $\sigma = \sigma^*_{1/2} = \sigma^*_{-1/2}$ and, with

$$\sqrt{|m|}\sigma = \sigma_1 = \sum_{d|m} d.$$

have our claim for $m \neq 0$.
4. We still have to treat the case $m = 0$, i.e., starting with

$$\begin{aligned} c_0(v,s) :=& v^s\zeta^*(2s) + v^{1-s}\zeta^*(2s-1), \\ \check{c}_0(v,s) :=& \partial_v c_0(v,s) = sv^{s-1}\zeta^*(2s) + (1-s)v^{-s}\zeta^*(2s-1), \end{aligned}$$

determine $\check{c}_0(v,1)$ and $\check{c}_0'(v,1)$.

We look at

$$a_1(s) := sv^{s-1}\zeta^*(2s) = \zeta(1-2s)\Gamma(1/2-s)\pi^{s-1/2}sv^{s-1}$$

and get, using standard material assembled in the Zeta Tool Remarks below,

$$\begin{aligned} a_1(1) &= -\zeta(-1)2\pi = \pi/6, \\ a_1'(s) &= (-2\zeta'(1-2s)/\zeta(1-2s) - \Gamma'(1/2-s)/\Gamma(1/2-s) + \log\pi + 1/s + \log v)a_1(s) \\ a_1'(1) &= (-2\zeta'(-1)/\zeta(-1) - \Gamma'(-1/2)/\Gamma(-1/2) + 1 + \log(\pi v))a_1(1) \\ &= (\pi/6)(24\zeta'(-1) + \gamma - 1 + \log(4\pi v)). \end{aligned}$$

Similarly, we take

$$a_2(s) := (1-s)v^{-s}\zeta^*(2s-1) = (1-s)\zeta(2s-1)\Gamma(s-1/2)\pi^{1/2-s}v^{-s}$$

and for

$$F(s) := (1-s)\zeta(2s-1) = (1-s)(1/(2(s-1)) + \gamma + \dots)$$

get

$$F(1) := -1/2,\ F'(1) = -\gamma,$$

while for

$$G(s) := \Gamma(s - 1/2)\pi^{1/2-s}v^{-s}$$

one has

$$\begin{aligned} G(1) &= \Gamma(1/2)\pi^{-1/2}v^{-1} = 1/v \\ G'(s) &= (\Gamma'(s-1/2)/\Gamma(s-1/2) - \log\pi - \log v)G(s) \\ G'(1) &= (\Gamma'(1/2)/\Gamma(1/2) - \log(\pi v))1/v \\ &= (1/v)(-\gamma - \log(4\pi v)), \end{aligned}$$

and, hence,

$$\begin{aligned} a_2(1) &= F(1)G(1) = -1/(2v) = \zeta(-1)4\pi(3/(2\pi v)) \\ a_2'(1) &= F'(1)G(1) + F(1)G'(1) = -\gamma(1/v) - (1/2)((-\gamma - \log(4\pi v))1/v \\ &= (1/2v)(-\gamma + \log(4\pi v)). \end{aligned}$$

Finally we have the claim for the constant terms

$$\begin{aligned} \check{c}_0(v,1) &= a_1(1) + a_2(1) = \zeta(-1)(-2\pi + 6/v) = \pi/6 - (1/2v) \\ \check{c}_0'(v,1) &= a_1'(1) + a_2'(1) \\ &= (\pi/6)(24\zeta'(-1) + \gamma - 1 + \log(4\pi v)) + (1/2v)(-\gamma + \log(4\pi v)). \end{aligned}$$

□

References

[AS] Abramovitz, M., Stegun, I.A.: *Handbook of mathematical functions.* US National bureau of standards 1972.

[BF] Bruinier, J.H., Funke, J.: *Traces of CM values of modular functions.* J. reine angew. Math. **594** (2006) 1-33.

[BK1] Berndt, R., Kühn, U. : *On Kudla's Green function for signature (2,2), part I* arXiv:1205.6417, Preprint 2012.

[BBK] Bruinier, J.H., Burgos Gil, J.I., Kühn, U.: *Borcherds Products and Arithmetic Intersection Theory on Hilbert Modular Surfaces.*Duke Math. J. **139** (2007) 1-88.

[BKK] Burgos Gil, J.I., Kramer, J., Kühn, U.: *Cohomogical arithmetic Chow rings.* J. Inst. Math. Jussieu **6** (2007) 1-172.

[FB] Freitag, E.; Busam, R.: *Funktionentheorie.* Springer-Verlag 1993.

[Iw] Iwaniec, H.: *Spectral Methods of Automorphic Forms.* GSM 53, AMS 2002.

[Kn] Knapp, A.W.: *Elliptic Curves.* Princeton University Press 1992.

[Ku] Kudla, S.: *Special Cycles and derivatives of Eisenstein series.* p. 243-270 in *Heegner Points and Rankin L-Series.* Math. Sci. Res. Inst. Publ. **49** Cambridge Univ. Press, Cambridge, 2004.

[KM1] Kudla, S., Millson, J.: *The theta correspondence and harmonic forms I.* Math. Ann. **247** (1986) 353-378.

[KM2] Kudla, S., Millson, J.: *The theta correspondence and harmonic forms II.* Math. Ann. **277** (1987) 267-314.

[KM3] Kudla, S., Millson, J.: *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables.* IHES Publ. Math.. **71** (1990) 121-172.

[KRY] Kudla, S., Rapoport, M., Yang, T.: *Derivatives of Eisenstein series and Faltings heights.* Compositio Math. **140** (2004) 289-351.

[KRY1] Kudla, S., Rapoport, M., Yang, T.: *Modular Forms and Special Cycles on Shimura Curves.* AM **161** Princeton 2006.

[K] Kühn, U.: *Generalized arithmetic intersection numbers.* J. reine angew. Math. **534** (2001) 209-236.

[La] lang, S.: *Elliptic Functions (second edition).* Springer 1987.

[Og] Ogg, A.: *Modular forms and Dirichlet Series.* Benjamin, New York 1969.

[Ra] Rankin, R.A.: *Modular Forms and Functions.* Cambridge Univerity Press, 1977

[S1] Siegel, C. L.: *Über die Zetafunktionen indefiniter quadratischer Formen II.* Math. Z. **44** (1939) 398-426.

[S2] Siegel, C. L.: *Indefinite quadratische Formen und Funktionentheorie I.* Math. Ann. **124** (1951) 17-54.

[Za] Zagier, D.: *Introduction to modular forms.* in *The Rankin-Selberg method for automorphic functions which are not of rapid decay.* J. Fac. Sci. Univ. Tokyo, Sect. I A **28** (1981) 415-437.